Physarum Can Compute Shortest Paths

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Abstract

Physarum Polycephalum is a slime mold that apparently is able to solve shortest path problems. A mathematical model has been proposed by biologists to describe the feedback mechanism used by the slime mold to adapt its tubular channels while foraging two food sources s_0 and s_1. We prove that, under this model, the mass of the mold will eventually converge to the shortest s_0-s_1 path of the network that the mold lies on, independently of the structure of the network or of the initial mass distribution.

This matches the experimental observations by the biologists and can be seen as an example of a "natural algorithm", that is, an algorithm developed by evolution over millions of years.

1 Introduction

Physarum Polycephalum is a slime mold that apparently is able to solve shortest path problems. Nakagaki, Yamada, and Tóth [NYT00] report about the following experiment; see Figure 1. They built a maze, covered it by pieces of Physarum (the slime can be cut into pieces which will reunite if brought into vicinity), and then fed the slime with oatmeal at two locations. After a few hours the slime retracted to a path that follows the shortest path in the maze connecting the food sources. The authors report that they repeated the experiment with different mazes; in all experiments, Physarum retracted to the shortest path. There are several videos available on the web that show the mold in action [you].

The paper [TKN07] proposes a mathematical model for the behavior of the slime and argues extensively that the model is adequate. We will not repeat the discussion here but only define the model. Physarum is modeled as an electrical network with time varying resistors. We have a simple undirected graph $G = (N, E)$ with distinguished nodes s_0 and s_1 modeling the food sources. Each edge $e \in E$ has a positive length L_e and a positive diameter $D_e(t)$; L_e is fixed, but $D_e(t)$ is a function of time. The resistance $R_e(t)$ of e is $R_e(t) = L_e/D_e(t)$. We force a current of value 1 from s_0 to s_1. Let $Q_e(t)$ be the resulting current over edge $e = (u, v)$, where (u, v) is an arbitrary orientation of e. The diameter of any edge e evolves according to the equation

$$\dot{D}_e(t) = |Q_e(t)| - D_e(t), \quad (1.1)$$

where $\dot{D}_e$ is the derivative of D_e with respect to time. In equilibrium ($\dot{D}_e = 0$ for all e), the flow through any edge is equal to its diameter. In non-equilibrium, the diameter grows (shrinks) if the absolute value of the flow is larger (smaller) than the diameter. In the sequel, we will mostly drop the argument t as is customary in the treatment of dynamical systems.

The model is readily turned into a computer sim-

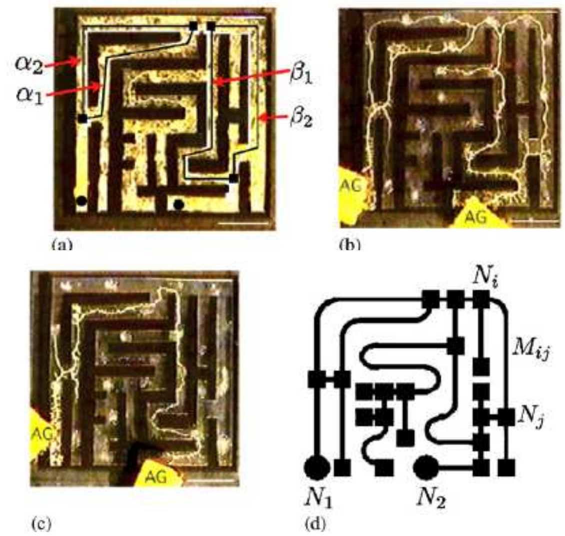

Figure 1: The experiment in [NYT00] (reprinted from there): (a) shows the maze uniformly covered by Physarum; yellow color indicates presence of Physarum. Food (oatmeal) is provided at the locations labelled AG. After a while the mold retracts to the shortest path connecting the food sources as shown in (b) and (c). (d) shows the underlying abstract graph. The video [you] shows the experiment.

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ulation. In an electrical network every vertex v has a potential p_v; p_v is a function of time. We may fix p_{s_1} to zero. For an edge $e = (u, v)$, the flow across e is given by $(p_u - p_v)/R_e$. We have flow conservation in every vertex except for s_0 and s_1, we inject one unit at s_0 and remove one unit at s_1. Thus

$$b_v = \sum_{u \in \delta(v)} \frac{p_v - p_u}{R_{uv}}$$

where $\delta(v)$ is the set of neighbors of v and $b_{s_0} = 1$, $b_{s_1} = -1$ and $b_v = 0$ otherwise. The node potentials can be computed by solving a linear system (either directly or iteratively). Tero et al. [TKN07] were the first to perform simulations of the model. They report that the network always converges to the shortest s_0-s_1 path, i.e., the diameters of the edges on the shortest path converge to one and the diameters on the edges outside the shortest path converge to zero. This holds true for any initial condition and assumes uniqueness of the shortest path.

Miyaji and Ohnishi [MO07, MO08] initiated the analytical investigation of the model. They argued convergence against the shortest path if G is a planar graph and s_0 and s_1 lie on the same face in some embedding of G.

2 Our Results

Our main result is a convergence proof for all graphs. For a network $G = (V, E, s_0, s_1, L)$, where $(L_e)_{e \in E}$ is a positive length function on the edges of G, we use $G_0 = (V, E_0)$ to denote the subgraph of all shortest source-sink paths, L^* to denote the length of a shortest source-sink path, and $\mathcal{E}^*$ to denote the set of all source-sink flows of value one in G_0. If we define the cost of flow Q as $\sum_e L_e Q_e$, then $\mathcal{E}^*$ is the set of minimum cost source-sink flows of value one. If the shortest source-sink path is unique, $\mathcal{E}^*$ is a singleton. The dynamics is *attracted* by a set A, if the distance (measured in any L_p-norm) between $D(t)$ and A converges to zero.

THEOREM 2.1. *Let $G = (V, E, s_0, s_1, L)$ be an undirected network with positive length function $(L_e)_{e \in E}$. Let $D_e(0) > 0$ be the diameter of edge e at time zero. The dynamics (1.1) are attracted to $\mathcal{E}^*$. If the shortest source-sink path is unique, the dynamics converges to the flow of value one along the shortest source-sink path.*

We conjecture that the dynamics converges to an element of $\mathcal{E}^*$, but only show attraction to $\mathcal{E}^*$. A key part of our proof is to show that the function

$$V = \frac{1}{\min_{S \in \mathcal{C}} C_S} \sum_{e \in E} L_e D_e + (C_{\{s_0\}} - 1)^2 \qquad (2.2)$$

decreases along all trajectories starting in a non-equilibrium configuration. Here, $\mathcal{C}$ is the set of all s_0-s_1 cuts, i.e., the set of all $S \subseteq N$ with $s_0 \in S$ and $s_1 \notin S$, $C_S = \sum_{e \in \delta(S)} D_e$ is the capacity of the cut S when the capacity of edge e is set to D_e, and $\min_{S \in \mathcal{C}} C_S$ (also abbreviated by C) is the capacity of the minimum cut. The first term in the definition of V is the normalized hardware cost; for any edge, the product of its length and its diameter may be interpreted as the hardware cost of the edge; the normalization is by the capacity of the minimum cut. We show that the first term decreases except when the maximum flow F in the network with capacities D_e is unique and moreover $|Q_e| = |F_e|/C$ for all e. The second term decreases as long as the capacity of the cut defined by s_0 is different from 1. We show that the capacity of the minimum cut converges to one. We infer from the decrease of V along all trajectories that $|D_e - |Q_e||$ converges to zero for all e. In the next step, we show that the potential difference $\Delta = p_{s_0} - p_{s_1}$ between source and sink converges to the length L^* of a shortest-source sink path. We use this to conclude that D_e and Q_e converge to zero for any edge $e \notin E_0$. Finally, we show that the dynamics is attracted by $\mathcal{E}^*$. We found the function V by a mixture of analytical investigation of a network of parallel links (see the full version of this article [BMV11]), extensive computer simulations, and guessing. Functions decreasing along all trajectories are called Lyapunov functions in dynamical systems theory [HS74]. Observe that the system (1.1) is defined by a vector field that is not continuously differentiable. Also, the function V is not everywhere differentiable. This introduces some technical difficulties.

The direction of the flow across an edge depends on the initial conditions and time. We do not know whether flow directions can change infinitely often or whether they become ultimately fixed. Under the assumption that flow directions stabilize, we can characterize the (late stages of the) convergence process. An edge $e = \{u, v\}$ becomes *horizontal* if $\lim_{t \to \infty} |p_u - p_v| = 0$ and it becomes *directed* from u to v (directed from v to u) if $p_u > p_v$ for all large t ($p_v > p_u$ for all large t). An edge *stabilizes* if it either becomes horizontal or directed and a network *stabilizes* if all its edges stabilize. If a network stabilizes, we partition its edges into a set E_h of horizontal edges and a set $\overrightarrow{E}$ of directed edges. If $\{u, v\}$ becomes directed from u to v, then $(u, v) \in \overrightarrow{E}$.

We introduce the notion of a *decay rate*. Let $r \leq 0$. A *quantity $D(t)$ decays with rate at least r*, if for every $\varepsilon > 0$ there is a constant A such that $\ln D(t) \leq A + (r + \varepsilon)t$ for all t. A *quantity $D(t)$ decays with rate at most r*, if for every $\varepsilon > 0$ there is a constant a such that $\ln D(t) \geq a + (r - \varepsilon)t$ for all t. A quantity

$D(t)$ decays with rate r, if it decays with rate at least and at most r.

LEMMA 2.1. *For $e \in E_h$, D_e decays with rate -1 and $|Q_e|$ decays with rate at least -1.*

We define a decomposition of G into paths P_0 to P_k, an orientation of these paths, a slope $f(P_i)$ for each P_i, a vertex labelling p^*, and an edge labelling r. P_0 is a[1] shortest s_0-s_1 path in G, $f(P_0) = 1$, $r_e = f(P_0) - 1$ for all $e \in P_0$, and $p^*_v = \text{dist}(v, s_1)$ for all $v \in P_0$, where $\text{dist}(v, s_1)$ is the shortest path distance from v to s_1. For $1 \le i \le k$, we have[2] $P_i = \text{argmax}_{P \in \mathcal{P}} f(P)$, where $\mathcal{P}$ is the set of all paths P in G with the following properties: (1) the startpoint a and the endpoint b of P lie on $P_0 \cup \ldots \cup P_{i-1}$, $p^*_a \ge p^*_b$, and $f(P) = (p^*_a - p^*_b)/L(P)$; (2) no interior vertex of P lies on $P_0 \cup \ldots \cup P_{i-1}$; and (3) no edge of P belongs to $P_0 \cup \ldots \cup P_{i-1}$. If $p^*_a > p^*_b$, we direct P_i from a to b. If $p^*_a = p^*_b$, we leave the edges in P_i undirected. We set $r_e = f(P_i) - 1$ for all edges of P_i, and $p^*_v = p^*_b + f(P_i)\,\text{dist}_{P_i}(v, b)$ for every interior vertex v of P_i. Figure 2(a) illustrates the path decomposition.

LEMMA 2.2. *There is an $i_0 \le k$ such that*

$$f(P_0) > f(P_1) > \ldots > f(P_{i_0}) > \\ > 0 = f(P_{i_0+1}) = \ldots = f(P_k).$$

THEOREM 2.2. *If a network stabilizes, $\overrightarrow{E} = \cup_{i \le i_0} E(P_i)$, the orientation of any edge $e \in \overrightarrow{E}$ agrees with the orientation induced by the path decomposition, and $E_h = \cup_{i > i_0} E(P_i)$. The potential of each node v converges to p^*_v. The diameter of each edge $e \in E \setminus P_0$ decays with rate r_e.*

We cannot prove that flow directions stabilize in general. For series-parallel graphs flow directions trivially stabilize. The Wheatstone graph shown in Figure 2(b) is the simplest graph, where flow directions may change over time.

THEOREM 2.3. *The Wheatstone graph stabilizes.*

The uncapacitated transportation problem generalizes the shortest path problem. With each vertex v a supply/demand b_v is associated. It is assumed that $\sum_v b_v = 0$. Nodes with positive b_v are called supply nodes and nodes with negative b_v are called demand nodes. In the shortest path problem, exactly two vertices have non-zero supply/demand. A feasible solution to the transportation problem is a flow f satisfying the mass balance constraints, i.e., for every vertex v, b_v is equal to the net flow out of v. The cost of a solution is $\sum_e L_e f_e$. The Physarum solver for the transportation problem is as follows: At any fixed time the currents $(Q_e)_{e \in E}$ are a feasible solution to the transportation problem which also satisfies Ohm's law. The dynamics evolve according to (1.1). The equilibria, i.e., $|Q_e| = D_e$ for all e, are precisely the flows with the equal-length property. Orient the edges in the direction of Q and drop the edges of flow zero. In the resulting graph, any two distinct directed paths with the same source and sink have the same length. Let $\mathcal{E}$ be the set of equilibria.

THEOREM 2.4. *The dynamics (1.1) is attracted to the set of equilibria $\mathcal{E}$. If any two equilibria have distinct cost, the dynamics converge to an optimum solution of the transportation problem.*

Theorem 2.1 is stronger in two respects. There we show attraction to the set of equilibria of minimum cost (now only to the set of equilibria) and convergence to the optimum solution if the optimum solution is unique (now only if no two equilibria have the same cost).

3 Related Work

Miyaji and Ohnishi [MO07, MO08] initiated the analytical investigation of the model. They argued convergence against the shortest path if G is a planar graph and s_0 and s_1 lie on the same face in some embedding of G. Ito et al. [IJNT11] study the dynamics (1.1) in a directed graph $G = (V, E)$; they do not claim that the model is justified on biological grounds. Each directed edge e has a diameter D_e. Let U be the underlying undirected graph. The conductivity of an undirected edge $\{u, v\}$ is the sum of the conductivities of the edges uv and vu (if both exist). The node potentials and flows in U are defined as above. The dynamics for the diameter of the directed edge uv is then $\dot{D}_{uv} = Q_{uv} - D_{uv}$. The dynamics of this model is very different from the dynamics of our model. The flow Q_{uv} may be positive and large because D_{vu} is large. The dynamics will increase D_{uv} (if present) and decrease D_{vu}. The model is simpler to analyze. They prove that the directed model is able to solve transportation problems.

4 Discussion and Open Problems

Physarum may be seen as an example of a natural computer, i.e., a computer developed by evolution over millions of years. It apparently can do more than computing shortest paths and solving transportation problems. In [TTS+10] the computational capabilities of Physarum are applied to network design and it is shown in lab and computer experiments that Physarum

[1] We assume that P_0 is unique.

[2] We assume that P_i is unique except if $f(P_i) = 0$.

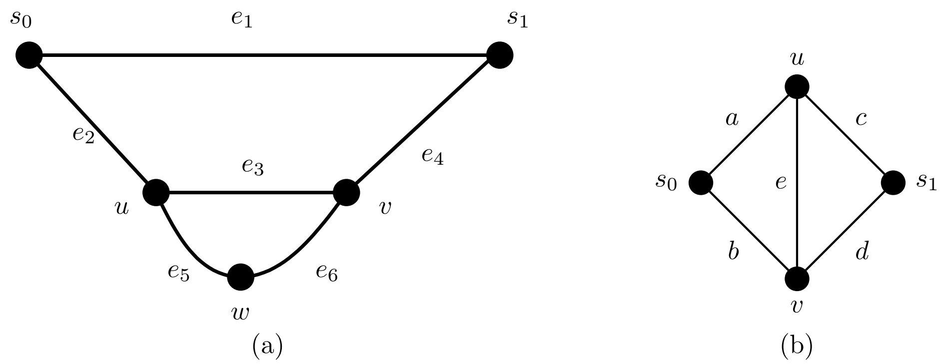

Figure 2: Part (a) illustrates the path decomposition. All edges are assumed to have length 1; $P_0 = (e_1)$, $P_1 = (e_2, e_3, e_4)$, $P_2 = (e_5, e_6)$, $p^*_{s_0} = 1$, $p^*_{s_1} = 0$, $p^*_v = 1/3$, $p^*_u = 2/3$, $p^*_w = 1/2$, $f(P_1) = 1/3$, and $f(P_2) = 1/6$. Part (b) shows the Wheatstone graph. The direction of the flow on edge $\{u, v\}$ may change over time; the flow on all other edges is always from left to right.

can compute approximate Steiner trees. No theoretical analysis is available. The book [Ada10] and the tutorial [NTK+09] contain many illustrative examples of the computational power of this slime mold.

Chazelle [Cha09] advocates the study of natural algorithms; i.e., "algorithms developed by evolution over millions of years", using computer science techniques. Traditionally, the analysis of such algorithms was the domain of biology, systems theory, and physics. Computer science brings new tools. For example, in our analysis, we crucially use the max-flow min-cut theorem. Natural algorithms can also give inspiration for the development of new combinatorial algorithms. A good example is [CKM+11], where electrical flows are essential for an approximation algorithm for undirected network flow.

We have only started the theoretical investigation of Physarum computation. Many interesting questions are open. We prove convergence for the dynamics $\dot{D}_e = f(|Q_e|) - D_e$ where f is the identity function. The biological literature also suggests the use of $f(x) = x^\gamma/(1+x)$ for some parameter γ. Can one prove convergence for other functions f? We prove that flow directions stabilize in the Wheatstone graph. Do they stabilize in general? We prove, but only for stabilizing networks, that the diameters of edges not on the shortest path converge to zero exponentially for large t. What can be said about the initial stages of the process? The Physarum computation is fully distributed; node potentials depend only on the potentials of the neighbors, currents are determined by potential differences of edge endpoints, and the update rule for edge diameters is local. Can the Physarum computation be used as the basis for an efficient distributed shortest path algorithm? What other problems can be provably solved with Physarum computations?

5 The Convergence Proof for the Shortest Path Problem

For the convergence proof, we will use some fundamental principles from the theory of electrical networks as they can be found, for example, in [Bol98, Chapters II, IX]. We start from the following simple facts. Recall that $\mathcal{C}$ is the set of s_0-s_1 cuts and $C_S = \sum_{e \in \delta(S)} D_e$. Also, let $L_{\min} = \min_e L_e$, $L_{\max} = \max_e L_e$, $n = |N|$, $m = |E|$.

Lemma 5.1. *The following hold for any edge $e \in E$ and any cut $S \in \mathcal{C}$:*

(i) $|Q_e| \le 1$.

(ii) $\sum_{e \in \delta(\{s_0\})} |Q_e| = 1$.

(iii) $D_e(t) \ge D_e(0) \exp(-t)$ *for all* t,

(iv) $D_e(t) \le 1 + (D_e(0) - 1)\exp(-t)$ *for all* t.

(v) $R_e \ge L_{\min}/2$ *for all sufficiently large* t.

(vi) $C_S(t) \ge 1 + (C_S(0) - 1)\exp(-t)$ *for all* t, *with equality if* $S = \{s_0\}$.

(vii) $C_{\{s_0\}} \to 1$ *as* $t \to \infty$.

(viii) Orient the edges according to the direction of the flow. For sufficiently large t, *there is a directed source-sink path of edges of diameter at least* $1/2m$.

(ix) $|\Delta_e| \le 2nmL_{\max}$ *for all sufficiently large* t.

(x) $\dot{D}_e/D_e \in [-1, 2nmL_{\max}/L_{\min}]$ *for all sufficiently large* t.

We will prove convergence for general graphs. In the following we will always assume that t is large enough that all the claims of Lemma 5.1 requiring a sufficiently large t hold. Recall that $D \in \mathbb{R}^E_+$ is an *equilibrium point* when $\dot{D}_e = 0$ for all $e \in E$, which is equivalent to $D_e = |Q_e|$ for all $e \in E$.

LEMMA 5.2. *At an equilibrium point,* $\min_{S\in\mathcal{C}} C_S = C_{\{s_0\}} = 1$.

Proof.

$$1 \le \min_{S\in\mathcal{C}} \sum_{e\in\delta(S)} |Q_e| = \min_{S\in\mathcal{C}} C_S \le$$
$$\le C_{\{s_0\}} = \sum_{e\in\delta(\{s_0\})} |Q_e| = 1.$$

LEMMA 5.3. *The equilibria are precisely the flows of value 1 in which all source-sink paths have the same length. If no two source-sink paths have the same length, the equilibria are precisely the simple source-sink paths.*

Proof. Let Q be a flow of value 1 in which all source-sink paths have the same length. We orient the edges such that $Q_e \ge 0$ for all e and show that $D = Q$ is an equilibrium point. Let E_1 be the set of edges carrying positive flow and let V_1 be the set of vertices lying on a source-sink path consisting of edges in E_1. For $v \in V_1$, set its potential to the length of the paths from v to s_1 in (V_1, E_1); observe that all such paths have the same length by assumption. Let Q' be the electrical flow induced by the potentials and edge diameters. For any edge $e = (u, v) \in E_1$ we have $Q'_e = D_e\Delta_e/L_e = D_e = Q_e$. Thus $Q' = Q$. For any edge $e \notin E_1$, we have $Q_e = 0 = D_e$. We conclude that D is an equilibrium point.

Let D be an equilibrium point and let Q_e be the corresponding current along edge e, where we orient the edges so that $Q_e \ge 0$ for all $e \in E$. Whenever $D_e > 0$, we have $\Delta_e = Q_e L_e/D_e = L_e$ because of the equilibrium condition. Since all directed s_0-s_1 paths span the same potential difference, all directed path from s_0 to s_1 in $\{e \in E : D_e > 0\}$ have the same length. Moreover, by Lemma 5.2, $\min_S C_S = 1$. Thus D is a flow of value 1.

LEMMA 5.4. *Let* $W = (C_{\{s_0\}} - 1)^2$. *Then* $\dot{W} = -2W \le 0$, *with equality iff* $C_{\{s_0\}} = 1$.

The following functions play a crucial role. Let $C = \min_{S\in\mathcal{C}} C_S$ and

$$V_S = \frac{1}{C_S} \sum_{e\in E} L_e D_e \qquad \text{for each } S \in \mathcal{C},$$
$$V = \max_{S\in\mathcal{C}} V_S + W,$$
$$h = -\frac{1}{C} \sum_{e\in E} R_e |Q_e| D_e + \frac{1}{C^2} \sum_{e\in E} R_e D_e^2.$$

LEMMA 5.5. *Let S be a minimum capacity cut at time t. Then* $\dot{V}_S(t) \le -h(t)$.

Proof. Let X be the characteristic vector of $\delta(S)$, that is, $X_e = 1$ if $e \in \delta(S)$ and 0 otherwise. Observe that $C_S = C$ since S is a minimum capacity cut. We have

$$\begin{aligned}
\dot{V}_S &= \sum_e \frac{\partial V_S}{\partial D_e} \dot{D}_e \\
&= \sum_e \frac{1}{C^2} \left(L_e C - \sum_{e'} L_{e'} D_{e'} X_e \right) (|Q_e| - D_e) \\
&= \frac{1}{C} \sum_e L_e |Q_e| - \frac{1}{C^2} \left(\sum_{e'} L_{e'} D_{e'} \right) \left(\sum_e X_e |Q_e| \right) + \\
&\qquad - \frac{1}{C} \sum_e L_e D_e + \frac{1}{C^2} \left(\sum_{e'} L_{e'} D_{e'} \right) \left(\sum_e X_e D_e \right) \\
&\le \frac{1}{C} \sum_e R_e |Q_e| D_e - \frac{1}{C^2} \sum_e R_e D_e^2 + \\
&\qquad - \frac{1}{C} \sum_e L_e D_e + \frac{1}{C} \sum_e L_e D_e \\
&= -h.
\end{aligned}$$

The only inequality follows from $L_e = R_e D_e$ and $\sum_e X_e |Q_e| \ge 1$, which holds because at least a unit current must cross S.

LEMMA 5.6. *Let* $f(t) = \max_{S\in\mathcal{C}} f_S(t)$ *where each f_S is continuous and differentiable. If $\dot{f}(t)$ exists, then there is $S \in \mathcal{C}$ such that $f(t) = f_S(t)$ and $\dot{f}(t) = \dot{f}_S(t)$.*

LEMMA 5.7. *$\dot{V}$ exists almost everywhere. If $\dot{V}(t)$ exists, then $\dot{V}(t) \le -h(t) - 2W(t) \le 0$, and $\dot{V}(t) = 0$ iff $\forall e,\ \dot{D}_e(t) = 0$.*

Proof. V is Lipschitz-continuous, since it is the maximum of a finite set of continuously differentiable functions. Since V is Lipschitz-continuous, the set of t's where $\dot{V}(t)$ does not exist has zero Lebesgue measure (see for example [CLSW98, Ch. 3], [MN92, Ch. 3]). When $\dot{V}(t)$ exists, one has $\dot{V}(t) = \dot{W}(t) + \dot{V}_S(t)$ for some S of minimum capacity (Lemma 5.6). Then $\dot{V}(t) \le -h(t) - 2W(t)$ by Lemmas 5.4 and 5.5.

The fact that $W \geq 0$ is clear. We now show that $h \geq 0$. To this end, let F represent a maximum s_0-s_1 flow in an auxiliary network having the same structure as G and where the capacity on edge e is set equal to D_e. In other words, F is an s_0-s_1 flow satisfying $|F_e| \leq D_e$ for all $e \in E$ and having maximum value. By the max-flow min-cut theorem, this maximum value is equal to $C = \min_{S \in \mathcal{C}} C_S$. But then,

$$
\begin{aligned}
-h &= \frac{1}{C}\sum_e R_e|Q_e|D_e - \frac{1}{C^2}\sum_e R_e D_e^2 \\
&\leq \frac{1}{C}\left(\sum_e R_e Q_e^2\right)^{1/2}\left(\sum_e R_e D_e^2\right)^{1/2} - \frac{1}{C^2}\sum_e R_e D_e^2 \\
&\leq \frac{1}{C}\left(\sum_e R_e \frac{F_e^2}{C^2}\right)^{1/2}\left(\sum_e R_e D_e^2\right)^{1/2} - \frac{1}{C^2}\sum_e R_e D_e^2 \\
&\leq \frac{1}{C^2}\left(\sum_e R_e D_e^2\right)^{1/2}\left(\sum_e R_e D_e^2\right)^{1/2} - \frac{1}{C^2}\sum_e R_e D_e^2 \\
&= 0,
\end{aligned}
$$

where we used the following inequalities:

- the Cauchy-Schwarz inequality:

$$
\sum_e (R_e^{1/2}|Q_e|)(R_e^{1/2}D_e) \leq \leq (\sum_e R_e Q_e^2)^{1/2}(\sum_e R_e D_e^2)^{1/2};
$$

- Thomson's Principle [Bol98, Theorem IX.2]: Q is a minimum energy flow of unit value, F/C is a feasible flow of unit value and hence $\sum_e R_e Q_e^2 \leq \sum_e R_e (F_e/C)^2$;

- $|F_e| \leq D_e$ for all $e \in E$.

Finally, one can have $h = 0$ if and only if all the above inequalities are equalities, which implies that $|Q_e| = |F_e|/C = D_e/C$ for all e. And $W = 0$ iff $\sum_{e \in \delta(\{s_0\})} D_e = 1 = \sum_{e \in \delta(\{s_0\})} |Q_e|$. So $h = W = 0$ iff $|Q_e| = D_e$ for all e.

The next lemma is a necessary technicality.

Lemma 5.8. *The function $t \mapsto h(t)$ is Lipschitz-continuous.*

Lemma 5.9. *$|D_e - |Q_e||$ converges to zero for all $e \in E$.*

Proof. Consider again the function h. We claim $h \to 0$ as $t \to \infty$. If not, there is $\varepsilon > 0$ and an infinite unbounded sequence $t_1, t_2, \ldots$ such that $h(t_i) \geq \varepsilon$ for all i. Since h is Lipschitz-continuous (Lemma 5.8), there is δ such that $h(t_i + \delta') \geq h(t_i) - \varepsilon/2 \geq \varepsilon/2$ for all $\delta' \in [0, \delta]$ and all i. So by Lemma 5.7, $\dot{V}(t) \leq -h(t) \leq -\varepsilon/2$ for every t in $[t_i, t_i + \delta]$ (except possibly a zero measure set), meaning that V decreases by at least $\varepsilon\delta/2$ infinitely many times. But this is impossible since V is positive and nonincreasing.

Thus for any $\varepsilon > 0$, there is t_0 such that $h(t) \leq \varepsilon$ for all $t \geq t_0$. Then, recalling that $R_e \geq L_{\min}/2$ for all sufficiently large t (Lemma 5.1.v), we find

$$
\begin{aligned}
\sum_e \frac{L_{\min}}{2}\left(\frac{D_e}{C} - |Q_e|\right)^2 &\leq \sum_e R_e\left(\frac{D_e}{C} - |Q_e|\right)^2 \\
&= \frac{1}{C^2}\sum_e R_e D_e^2 + \sum_e R_e Q_e^2 - \frac{2}{C}\sum_e R_e|Q_e|D_e \\
&\leq \frac{2}{C^2}\sum_e R_e D_e^2 - \frac{2}{C}\sum_e R_e|Q_e|D_e \\
&= 2h \leq 2\varepsilon,
\end{aligned}
$$

where we used once more the inequality $\sum_e R_e Q_e^2 \leq \sum_e R_e D_e^2/C^2$, which was proved in Lemma 5.7. This implies that for each e, $D_e/C - |Q_e| \to 0$ as $t \to \infty$. Summing across $e \in \delta(\{s_0\})$, and using Lemma 5.1.ii, we obtain $C_{\{s_0\}}/C - 1 \to 0$ as $t \to \infty$. From Lemma 5.1, $C_{\{s_0\}} \to 1$ as $t \to \infty$, so $C \to 1$ as well.

To conclude, we show that $D_e/C - |Q_e| \to 0$ and $C \to 1$ together imply $D_e - |Q_e| \to 0$. Let $\varepsilon > 0$ be arbitrary. For all sufficiently large t, $|D_e/C - |Q_e|| \leq \varepsilon$, $|1 - C| \leq \varepsilon$, $D_e \leq 2$, and $C \geq 1/2$. Thus

$$
|D_e - |Q_e|| \leq |D_e - D_e/C| + |D_e/C - |Q_e|| \leq \leq D_e\frac{|C-1|}{C} + |D_e/C - |Q_e|| \leq 5\varepsilon.
$$

Lemma 5.10. *Let $\Delta = p_{s_0} - p_{s_1}$ be the potential difference between source and sink. Then Δ converges to the length L^* of a shortest source-sink path.*

Proof. Let $\mathcal{L}$ be the set of lengths of simple source-sink paths. We first show that Δ converges to a point in $\mathcal{L}$ and then show convergence to L^*.

Orient edges according to the direction of the flow. By Lemma 5.1.viii, there is a directed source-sink path P of edges of diameter at least $1/2m$. Let $\varepsilon > 0$ be arbitrary. We will show $|\Delta - L_P| \leq \varepsilon$. For this it suffices to show $|\Delta_e - L_e| \leq \varepsilon/n$ for any edge e of P, where Δ_e is the potential drop on e. By Ohm's law the potential drop on e is $\Delta_e = (Q_e/D_e)L_e$ and hence $|\Delta_e - L_e| = L_e|Q_e/D_e - 1| = L_e|(Q_e - D_e)/D_e| \leq 2mL_{\max}|Q_e - D_e|$. The claim follows since $|Q_e - D_e|$ converges to zero.

The set $\mathcal{L}$ is finite. Let ε be positive and smaller than half the minimal distance between two elements in $\mathcal{L}$. By the preceeding paragraph, there is for all sufficiently large t a path P_t such that $|\Delta - L_{P_t}| \leq \varepsilon$.

Since Δ is a continuous function of time, L_{P_t} must become constant. We have now shown that Δ converges to an element in $\mathcal{L}$.

We will next show that Δ converges to L^*. Assume otherwise and let P' be a shortest undirected source-sink path. Let $W_{P'} = \sum_{e \in P'} L_e \ln D_e$. This function was already used by Miyaji and Ohnishi [MO08]. We have

$$\dot{W}_{P'} = \sum_{e \in P'} \frac{L_e}{D_e}(|Q_e| - D_e) = \sum_{e \in P'} |\Delta_e| - \sum_{e \in P'} L_e \geq \\ \geq p_{s_0} - p_{s_1} - L_{P'} = \Delta - L^*.$$

Let $\delta > 0$ be such that there is no source-sink path with length in the open interval $(L^*, L^*+2\delta)$. Then $\Delta - L^* \geq \delta$ for all sufficently large t and hence $\dot{W}_{P'} \geq \delta$ for all sufficiently large t. Thus $W_{P'}$ goes to $+\infty$. However $W_{P'} \leq nL_{\max}$ for all sufficiently large t since $D_e \leq 2$ for all e and t large enough. This is a contradiction. Thus Δ converges to L^*.

LEMMA 5.11. *Let e be any edge that does not lie on a shortest source-sink path. Then D_e and Q_e converge to zero.*

Proof. Since $|D_e - |Q_e||$ converges to zero, it suffices to prove that Q_e converges to zero. Assume otherwise. Then there is a $\delta > 0$ such that $|Q_e| \geq \delta$ for arbitrarily large t.

Consider any such t and orient the edges according to the direction of the flow at time t. Let $e = (u, v)$. Because of flow conservation, there must be an edge into u and an edge out of v carrying flow at least Q_e/n. Continuing in this way, we obtain a source-sink path P in which every edge carries flow at least $Q_e/n^n \geq \delta/n^n$; P depends on time and $L_P > L^*$ always. We will show $|\Delta - L_P| \leq (L_P - L^*)/4$ for sufficiently large t, a contradiction to the fact that Δ converges to L^*. For this it suffices to show $|\Delta_g - L_g| \leq (L_P - L^*)/(4n)$ for any edge g of P, where Δ_g is the potential drop on g. By Ohm's law the potential drop on g is $\Delta_g = (Q_g/D_g)L_g$ and hence $|\Delta_g - L_g| = |Q_g/D_g - 1|L_g = |(Q_g - D_g)/D_g|L_g \leq L_{\max}|Q_g - D_g|/D_g$. For large enough t, $|Q_g - D_g| \leq \min(\delta/(2n^n), \delta(L_P - L^*)/(8n^{n+1}L_{\max}))$. Then $D_g \geq Q_g - |Q_g - D_g| \geq \delta/(2n^n)$ and hence $L_{\max}|Q_g - D_g|/D_g \leq (L_P - L^*)/(4n)$.

Recall that $\mathcal{E}^*$ is the set of flows of value one in the network of shortest source-sink paths.

THEOREM 5.1. *The dynamics are attracted by $\mathcal{E}^*$. If the shortest source-sink path is unique, the dynamics converges against the flow of value 1 on the shortest source-sink path.*

Proof. Q is a source-sink flow of value one at all times. We show first that Q is attracted to $\mathcal{E}^*$. Orient the edges in the direction of the flow. We can decompose Q into flowpaths. For an oriented path P, let 1_P be the unit flow along P. We can write $Q = \sum_P x_p 1_P$, where x_P is the flow along the path P. This decomposition is not unique. We group the flowpath into two sets, the paths running inside G_0 and the paths using an edge outside G_0, i.e.,

$$Q = Q_0 + Q_1, \text{ where } Q_0 = \sum_{P \text{ is a path in } G_0} x_P 1_P.$$

Q_0 is a flow in G_0 and each flowpath in Q_1 is a non-shortest source-sink path.[3] We show that the value of Q_0 converges to one.

Assume otherwise. Then there is a $\delta > 0$ such that the value of Q_1 is at least δ for arbitrarily large times t. At any such time there is an edge $e \notin E_0$ carrying flow at least δ/m; this holds since source-sink cuts contain at most m edges. Since there are only finitely many edges, there must be an edge $e \notin E_0$ for which Q_e does not converge to zero, a contradiction to Lemma 5.11.

We have now shown that the distance between Q and $\mathcal{E}^*$ converges to zero. By Lemma 5.9, $|D_e - |Q_e||$ converges to zero for all e and hence the distance between Q and D converges to zero. Thus D is attracted by $\mathcal{E}^*$.

Finally, if the shortest source-sink path is unique, $\mathcal{E}^*$ is a singleton and hence D converges to the flow of value one along the shortest source-sink path.

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[3]The decomposition into Q_0 and Q_1 can be constructed as follows. Initialize Q_0 to Q and Q_1 to the empty flow. Consider any edge $e \notin E_0$ carrying positive flow in Q_0, say ε. Let P an oriented source-sink path carrying ε units of flow and using e. Add $\varepsilon 1_P$ to Q_1 and subtract it from Q_0. Continue until Q_0 is a flow in G_0.

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