# A note on the construction of Sobolev almost periodic invariant tori for the 1d NLS

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**Abstract**

We announce a method for the construction of almost periodic solutions of the one dimensional analytic NLS with only Sobolev regularity both in time and space. This is the first result of this kind for PDEs.
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## 1 Overview and main result

In KAM theory for PDEs, one of the most challenging problems of the last twenty years is the construction of *almost-periodic* solutions[1]. Very few results are known in this direction, and all of them investigate models with *external parameters* in order to avoid resonances and focus on the small divisor problem. In particular, in all the existent literature, the construction of almost periodic solutions is achieved at the cost of an *extremely high* decay of their Fourier coefficients, which approach zero super-exponentially, exponentially or sub-exponentially (Gevrey). Indeed, because of the fact that the classical KAM procedure is not uniform in the dimension, one cannot naively construct a quasi-periodic solution supported on a $n$-dimensional invariant torus and then take the limit $n \to \infty$: one would fall on the elliptic fixed point. Relying on an NLS with multiplicative potential (producing an infinite set of free parameters) and smoothing nonlinearity, Pöschel (partially) tackles this problem in [27], by iteratively constructing its solutions though successive small perturbations of finite-dimensional tori, parametrized though action-angle variables and eventually characterized by a very strong compactness property: in order to overcome the dependence of the KAM estimates on their dimension, the radii of these tori have to shrink super-exponentially, this leading to very regular solutions. See also [20] for a generalization of Pöschel's approach to the analytic cathegory, by using Töplitz-Lipschitz function techniques.
Bourgain understood that when the dimension grows to infinity, action-angle variables become the Achilles' heel of the KAM procedure (they are not even well defined, in general) and in his pioneering work [13] on the quintic NLS with Fourier multipliers (providing external parameters in $\ell^\infty$), he proposed a different approach by working directly in cartesian coordinates, without introducing any action-angle variables, and relying on a Diophantine condition which is taylored for the infinite dimension. For most choices of the parameters, this lead to the construction of almost periodic invariant tori which support Gevrey solutions.

[1]i.e. solutions which are limit (in the uniform topology in time) of quasi-periodic solutions

The recent work [9] extends the techniques of [13] and proposes a novel, flexible approach which allows to construct in a unified framework, both maximal and elliptic invariant tori of any dimension which are the support of the desired Gevrey solutions, for an NLS with Fourier multipliers which does not necessarily preserve momentum. The persistence of the invariant tori is achieved though an abstract normal form theorem "à la Herman", whose estimates are *uniform in the dimension* (see [9, Theorems 3 and 7.1]). See also [6, 25, 26] for a survey on this technique.

The possibility of constructing almost-periodic solutions for a *fixed* PDE, i.e. "eliminating" the external parameters through amplitude-frequency modulation, appears to be intimately related to the regularity issues. Roughly speaking, a fast decay of the "actions" (needed for the scheme to converge) leads to a weak modulation of frequencies which in turn results in bad bounds on the small divisors. It then becomes fundamental to look for almost-periodic solutions in lower regularity spaces. However this appears to be a very difficult problem, due to the presence of extremely small divisors.

An analogous problem with rapidly vanishing small divisors arises in Birkhoff Normal Form theory for PDEs. Indeed in the analytic or Gevrey case one has sub-exponential stability times (see [18] and [15]), whereas in the Sobolev case the best known estimates have a power growth in the Sobolev exponent (see [2], [19], [5], [7]).

The counterpart of total and long time stability results is the construction of unstable trajectories, which undergo growth of the Sobolev norms, see [11, 14, 21–23].

In the context of quasi-periodic solutions there is a wide literature regarding solutions of finite regularity. However most of the interest is in the case of a non-linearity which is only Sobolev. The strategy is to apply a Nash-Moser scheme and prove tame estimates on the inverse of the linearized equation at an approximate solution. This method was proposed in [3] (generalizing the seminal works [17], [12] concerning the analytic case) via multiscale analysis, see also [4] or [1] for a reducibility approach. Of course one can apply this techniques also in the case of analytic non-linearities. Another possibility is to construct solutions which are analytic in time and only finite regularity in space (see, e.g. [24], [16]). Note however that solutions obtained with such methods are often actually smooth (by bootstrap arguments). Unfortunately it is not at all clear how (or even whether it is possible) to extend such ideas to almost-periodic solutions.

A main difference with the quasi-periodic case is that for almost-periodic solutions the topology of the phase space is crucial. Indeed, in the quasi-periodic case (at least for semi-linear PDEs) one typically looks for an *analytic embedded finite* dimensional torus in a fixed phase space of $x$-dependent functions, by modulating the analyticity strip in the angles. Then the analyticity in time directly follows. On the other hand, it is not obvious at all (and clearly it strongly depends on the topology of the phase space), whether the *embedded*[2] *infinite* dimensional torus is analytic. Anyway, the analyticity in time does not follow since the map $t \mapsto \omega t$ is not even continuous[3]. Generically, almost-periodic solutions are not continuous trajectories in the phase space. However, such solutions can be regular as complex valued functions of time and space, depending on the regularity of the phase space. See Remark 1.2.

Taking all the advantage from the flexible construction proposed in [9], we present here the very first result of persistence of almost-periodic solutions with finite regularity both in time and space. We stress that our solutions are not maximal tori but instead are mostly localized on a *sparce lattice*.

The core of our strategy is that in constructing solutions mostly localized in such lattices we may impose *very strong* Diophantine conditions, see Definition 1.2, so that our small divisors can be controlled similarly to the Gevrey case of [9]. The key points are the definition of diophantine vectors 1.2, the measure estimates of Theorem 2 and the bounds on the homological equation in Lemma 3.1 .

Let us consider families of NLS equations on the circle with external parameters of the form:

$$\mathrm{i}u_t + u_{xx} - V * u + f(|u|^2)u = 0\,. \tag{1.1}$$

Here $\mathrm{i} = \sqrt{-1}$, $u = \sum_{j\in\mathbb{Z}} u_j e^{\mathrm{i}jx}$, $V*$ is a Fourier multiplier

$$V * u = \sum_{j\in\mathbb{Z}} V_j u_j e^{\mathrm{i}jx}\,, \quad (V_j)_{j\in\mathbb{Z}} \in [-1/2, 1/2]^{\mathbb{Z}} \subset \ell^\infty(\mathbb{R}) \tag{1.2}$$

[2]In the infinite-dimensional case, even the definition of such an embedding requires accuracy in the choice of the topology. Indeed most of the existent literature bypass this issue and directly construct the solutions as limit of quasi-periodic ones.

[3]At least if $\sup_j |\omega_j| = \infty$ as it is typical in PDEs.

and $f(y)$ is real analytic in $y$ in a neighborhood of $y=0$. We shall assume that $f(0)=0$. By analyticity, for some $\mathtt{R}>0$ we have

$$f(y)=\sum_{d=1}^{\infty} f^{(d)} y^{d}, \quad |f|_{\mathtt{R}}:=\sum_{d=1}^{\infty}|f^{(d)}| \mathtt{R}^{d}<\infty, \tag{1.3}$$

We look for solutions in Fourier series $u(x)=\sum_{j\in\mathbb{Z}} u_j e^{\mathrm{i}jx}$, where $(u_j)_{j\in\mathbb{Z}}$ belongs to the scale of Banach spaces

$$\mathtt{w}_p:=\left\{u:=(u_j)_{j\in\mathbb{Z}}\in\ell^2(\mathbb{C}) \;:\; \|u\|_p:=\sup_{j\in\mathbb{Z}}|u_j|\lfloor j\rfloor^p<\infty\right\}, \quad p>1, \tag{1.4}$$

where[4] $\lfloor j\rfloor:=\max\{2,|j|\}$. As it is common habit, we endow $\mathtt{w}_p\subset\ell^2$ with the symplectic structure inherited from $\ell^2$. Note that $u(x)\in C^k$ for every $k<p-1$.

It is well known that (1.1) is a hamiltonian system with Hamiltonian

$$H_V(u):=\sum_{j\in\mathbb{Z}}(j^2+V_j)|u_j|^2+P, \quad \text{with} \quad P:=\int_{\mathbb{T}}F(x,|\sum_j u_j e^{\mathrm{i}jx}|^2)dx, \quad F(y):=\int_0^y f(s)ds. \tag{1.5}$$

**Definition 1.1** (Tangential sites)**.** Let $\mathcal{S}$ be any unbounded subset of $\mathbb{N}$. Let $s(i)$ be a smooth strictly increasing function[5] $s:[0,+\infty)\to[0,+\infty)$ such that $\mathcal{S}=s(\mathbb{N})$. We call $i(s)$ the inverse function of $s(i)$. Assume that there exist $i_*\ge 21$ and $1<\eta\le 2$ such that

$$s(i)\ge e^{(\log i)^{1+\eta}}, \qquad \forall i\ge i_*; \tag{1.6a}$$

$$s(i+i')\ge s(i)+s(i'), \qquad s(ji)\ge js(i), \qquad \forall\, j\ge 1, \quad i,i'\ge i_*; \tag{1.6b}$$

$$s(i^2)\ge s^2(i), \quad \forall i\ge i_*. \tag{1.6c}$$

We call tangential a site $s$ belonging to $\mathcal{S}$. We split $\mathbb{Z}=\mathcal{S}\sqcup\mathcal{S}^c$ and call normal a site $s$ belonging to $\mathcal{S}^c$.

Define the set of "tangential frequencies"

$$\mathcal{Q}_{\mathcal{S}}:=\left\{\nu=(\nu_j)_{j\in\mathcal{S}}\in\mathbb{R}^{\mathcal{S}} \;:\; \left|\nu_j-j^2\right|<\frac{1}{2}\right\}. \tag{1.7}$$

The main result is the following

**Theorem 1.** *Consider a translation invariant NLS Hamiltonian as in* (1.1)*. For any $p>1$, $\gamma>0$ there exists $\varepsilon_*=\varepsilon_*(p)>0$ such that, for all $r>0$ satisfying*

$$\varepsilon:=\frac{|f|_{\mathtt{R}}}{\gamma\mathtt{R}}r^2\le\varepsilon_* \tag{1.8}$$

*for every*

$$\sqrt{I}\in B_r(\mathtt{w}_p) \quad \text{with} \quad I_j=0 \quad \text{for} \quad j\in\mathcal{S}^c \tag{1.9}$$

*and for any $W\in[-1/4,1/4]^{\mathcal{S}^c}$ such that $W_0\neq 0$, the following holds.*
*There exist a positive measure Cantor-like set $\mathcal{C}\subset\mathcal{Q}_{\mathcal{S}}$, such that for all $\nu\in\mathcal{C}$ there exists a potential $V\in[-1/2,1/2]^{\mathbb{Z}}$ and a analytic change of variables $\Phi: B_{2r}(\mathtt{w}_p)\to B_{4r}(\mathtt{w}_p)$ such that*

$$\mathcal{T}_I:=\{u\in\mathtt{w}_p : |u_j|^2=I_j \quad \forall j\in\mathbb{Z}\} \tag{1.10}$$

*is an elliptic KAM torus of frequency $\nu$ for $H_V\circ\Phi$ with $V_j=W_j$ for $j\in\mathcal{S}^c$. Finally $V$ depends on $\nu$ in a Lipschitz way.*

[4] Obviously one could also take the more standard weight $\langle j\rangle:=\max\{1,|j|\}$, which generates the same Banach space. We made such choice for merely technical reasons.

[5] The function $s(i)$ is obviously not unique but its restriction to $\mathcal{S}$ is unique. The same holds for its inverse $i(s)$.

*Remark* 1.1. 1) If $\mathcal{S}_1, \mathcal{S}_2$ are admissible tangential sites (according to Definition 1.1) and $\mathcal{S}_0$ is any bounded subset of $\mathbb{Z}$, then we can consider also admissible tangential sites of the form $-\mathcal{S}_1$, $(-\mathcal{S}_1)\cup\mathcal{S}_0$, $\mathcal{S}_1\cup\mathcal{S}_0$, $(-\mathcal{S}_1)\cup\mathcal{S}_2\cup\mathcal{S}_0$.
2) The super-linearity assumptions (1.6b) and (1.6c) are essential for our estimate on the homological equation to work. The asymptotic growth in (1.6a) is only needed in the KAM step. A slower growth would give rise to a too large estimate on the solution of the homological equation which would not be compensated anymore by the quadratic convergence of the KAM scheme.
3) For example we can choose $s(i) = 2^i$ or a slower growth as $s(i) = [e^{\log^{1+\eta} i}]$ in Definition 1.1 ($[\cdot]$ being the integer part); note that the fact that $\mathcal{S}$ is sparse is essential. A proof of Theorem 1 in the particular case $s(i) = 2^i$ can be found in [10]. The complete proof in the general case together with other extensions will be discussed in the forthcoming paper [8].

*Remark* 1.2. Since we are able to construct solutions for every $\sqrt{I} \in \bar{B}_r(\mathtt{w}_p)$ with $I_j = 0$ for $j \in \mathcal{S}^c$ and we have that $|u_j(t)| \sim \sqrt{I}_j$, then most of our solutions have only Sobolev regularity (both in space and time) and are not analytic, Gevrey or even $C^\infty$. To be more explicit, the map

$$\imath : \mathbb{T}^\infty \to \mathcal{T}_I \subset \mathtt{w}_p, \quad \varphi = (\varphi_j)_{j\in\mathbb{Z}} \mapsto (\sqrt{I_j} e^{\mathrm{i}\varphi_j})_{j\in\mathbb{Z}}$$

is analytic provided that we endow $\mathbb{T}^\infty$ with the $\ell^\infty$-topology. On the other hand, the map $\psi : \mathbb{R} \to \mathbb{T}^\infty$, $t \mapsto \omega t$ is not even continuous. Of course the regularity of $\imath \circ \psi$ depends on the choice of the actions $I_j$. If we take for instance $I_j = \langle j\rangle^{-p}$, then $\imath\circ\psi : \mathbb{R} \to \mathtt{w}_p$ is not continuous. On the other hand, for all $x \in \mathbb{T}$, the map

$$t \mapsto \imath\circ\psi(t,x) = \sum_j \langle j\rangle^{-p} e^{\mathrm{i}(\omega_j t + jx)}$$

is $C^k(\mathbb{R},\mathbb{C})$ for all $k < \frac{p-1}{2}$.
We can be rather explicit in our description of the set $\mathcal{C}$. We start by fixing the hypercube

$$\mathcal{Q} := \left\{ \omega = (\omega_k)_k \in \mathbb{R}^{\mathbb{Z}} \,:\, \left|\omega_k - k^2\right| < \frac{1}{2} \right\}, \qquad \mathcal{Q} = \mathcal{Q}_{\mathcal{S}} \times \mathcal{Q}_{\mathcal{S}^c} \tag{1.11}$$

and by introducing the following

**Definition 1.2** (Diophantine condition)**.** Let $\tau > 1$. We say that a vector $\omega \in \mathcal{Q}$ belongs to $\mathtt{D}_{\gamma,\mathcal{S}}$ if it satisfies

$$|\omega\cdot\ell| \geq \gamma \prod_{s\in\mathcal{S}} \frac{1}{(1+|\ell_s|^2\langle i(s)\rangle^2)^\tau}\,, \quad \forall \ell : 0 < |\ell| < \infty\,, \quad \sum_{s\in\mathcal{S}^c} |\ell_s| \leq 2\,, \quad \pi(\ell) = \mathtt{m}(\ell) = 0\,, \tag{1.12}$$

where $i(s)$ is the inverse function of $s(i)$, $\pi(\ell) := \sum_{s\in\mathbb{Z}} s\ell_s$ is the "momentum" and $\mathtt{m}(\ell) := \sum_{s\in\mathbb{Z}} \ell_s$ is the "mass".

*Remark* 1.3. Note that (1.12) is a much stronger diophantine condition that the one proposed in [13] (or [9]), where the denominators were of the form $1 + |\ell_s|^2 s^2$.

**Theorem 2.** *Under the hypotheses of Theorem 1, there exist $C > 1$and a Lipschitz map*

$$\Omega : \mathcal{Q}_{\mathcal{S}} \to \mathcal{Q}_{\mathcal{S}^c}\,, \qquad |\Omega_j - j^2 - W_j| + \gamma|\Omega_j|^{\mathrm{Lip}} \leq C\gamma\varepsilon \quad \forall j \in \mathcal{S}^c\,, \tag{1.13}$$

*such that*

$$\mathcal{C} := \{\nu \in \mathcal{Q}_{\mathcal{S}} : \quad (\nu, \Omega(\nu)) \in \mathtt{D}_{\gamma,\mathcal{S}}\}$$

*where $|\cdot|^{\mathrm{Lip}}$ stands for the classical Lipschitz semi-norm. Moreover* $\mathrm{meas}(\mathcal{Q}_{\mathcal{S}} \setminus \mathcal{C}) \leq C\gamma$.

# 2 Functional Setting

Following [9] (see Definition 2.1 with $a = s = 0$) we introduce the space of regular Hamiltonians $\mathcal{H}_r(\mathtt{w}_p)$.

**Definition 2.1** (Regular Hamiltonians)**.** Consider a formal power series expansion

$$H(u) = \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{M}} H_{\boldsymbol{\alpha},\boldsymbol{\beta}} u^{\boldsymbol{\alpha}} \bar{u}^{\boldsymbol{\beta}} , \qquad u^{\boldsymbol{\alpha}} := \prod_{j\in\mathbb{Z}} u_j^{\boldsymbol{\alpha}_j} , \tag{2.1}$$

satisfying the reality condition

$$H_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \overline{H}_{\boldsymbol{\beta},\boldsymbol{\alpha}} , \qquad \forall\, (\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{M} , \tag{2.2}$$

where[6]

$$\mathcal{M} := \left\{ (\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathbb{N}^{\mathbb{Z}} \times \mathbb{N}^{\mathbb{Z}} , \ \text{ s..t. } \ |\boldsymbol{\alpha}| = |\boldsymbol{\beta}| , \ \sum_{j\in\mathbb{Z}} j(\boldsymbol{\alpha}_j - \boldsymbol{\beta}_j) = 0 \right\} \tag{2.3}$$

and here and in the following we always restrict to multi-indexes with finite norm $|\boldsymbol{\alpha}| := \sum_{j\in\mathbb{Z}} \boldsymbol{\alpha}_j$.
We denote by $\mathcal{H}_{r,p}$ for $p > 1$, $r > 0$ the space of regular Hamiltonians, i.e. those $H$ such that

$$|H|_{r,p} := \frac{1}{2} \sup_j \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{M}} |H_{\boldsymbol{\alpha},\boldsymbol{\beta}}| \big(\boldsymbol{\alpha}_j + \boldsymbol{\beta}_j\big) u_p^{\boldsymbol{\alpha}+\boldsymbol{\beta}-2e_j} < \infty , \tag{2.4}$$

where $u_p = u_p(r)$ is defined as

$$u_{p,j}(r) := r\lfloor j \rfloor^{-p} . \tag{2.5}$$

Note that $|\cdot|_{r,p}$ is a seminorm on $\mathcal{H}_{r,p}$ and a norm on its subspace

$$\mathcal{H}^0_{r,p} := \{ \ H \in \mathcal{H}_{r,p} \ \text{ with } \ H(0) = 0 \ \} , \tag{2.6}$$

endowing $\mathcal{H}^0_{r,p}$ with a Banach space structure.

To control the Lipschitz dependence on the frequency throughout the iterative scheme, we define the following weighted Lipschitz semi-norm. Fix $\gamma > 0$ and assume that $H = H(\omega) \in \mathcal{H}_{r,p}$ for every $\omega \in \mathtt{D}_{\gamma,\mathcal{S}}$. We then define the semi-norm (as usual $|v|_\infty := \sup_{j\in\mathbb{Z}} |v_j|$)

$$\|H\|_{r,p} \quad := \quad \sup_{\omega\in\mathtt{D}_{\gamma,\mathcal{S}}} |H(\omega)|_{r,p} + \gamma \sup_{\substack{\omega,\omega'\in\mathtt{D}_{\gamma,\mathcal{S}} \\ \omega\neq\omega'}} \frac{|H(\omega) - H(\omega')|_{r,p}}{|\omega - \omega'|_\infty} < \infty . \tag{2.7}$$

Finally we set

$$\mathcal{H}^{\rm Lip}_{r,p} := \Big\{ H(\omega) \in \mathcal{H}_{r,p}, \ \ \omega \in \mathtt{D}_{\gamma,\mathcal{S}} , \quad \text{with} \quad \|H\|_{r,p} < \infty \Big\} .$$

In [9] we show that the semi-norm $\|\cdot\|_{r,p}$ is monotone decreasing in $p$ and monotone increasing in $r$. Finally, it behaves well with respect to the Hamiltonian flows and Poisson brackets. See Proposition 2.1 and Lemma 2.1 of [7].

# 3 Strategy of the proof

The seminal idea contained in [13] for proving the persistence of a full dimensional invariant torus, consisted in smartly rewriting $H_{\rm V}$ in a way that one could select those terms preventing the torus to be invariant for its dynamics. This idea has been formalized in [9], in terms of a *degree decomposition* with increasing order of zero at $\mathcal{T}_I$, defined for any regular Hamiltonian. For convenience of the reader, we sketch here the main features needed to prove our persistence theorem, while for detailed statements and proofs we address the reader to [9, Section 4].
We want to prove that, in suitable variables, $\mathcal{T}_I$ introduced in (1.10) is an invariant torus on which the flow is linear with frequencies $\omega$. To this purpose we introduce a suitable degree decomposition, whose main idea is to

[6] Conditions $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$ and $\sum_{j\in\mathbb{Z}} j(\boldsymbol{\alpha}_j - \boldsymbol{\beta}_j) = 0$ correspond to mass momentum conservation, respectively.

make a *power series expansion centered at* $I$ without introducing a singularity in order to highlight the terms which prevent $\mathcal{T}_I$ to be invariant of frequency $\omega$. Consider a Hamiltonian $H(u)$ expanded in Taylor series at $u = 0$ and tautologically rewrite $H$ as

$$H = \sum_{\substack{m,\alpha,\beta\in\mathbb{N}^{\mathcal{S}} \\ \alpha\cap\beta=\emptyset \\ a,b\in\mathbb{N}^{\mathcal{S}^c}}} H_{m,\alpha,\beta,a,b} |v|^{2m} v^\alpha \bar{v}^\beta z^a \bar{z}^b \tag{3.1}$$

where, by slight abuse of notation[7], $u = (v, z)$ with $v = (v_j)_{j\in\mathcal{S}} := (u_j)_{j\in\mathcal{S}}$ and $z = (z_j)_{j\in\mathcal{S}^c} := (u_j)_{j\in\mathcal{S}^c}$. Then introduce the auxiliary "action" variables $w = (w_j)_{j\in\mathcal{S}}$ substituting $|v|^{2m} v^\alpha \bar{v}^\beta z^a \bar{z}^b \rightsquigarrow w^m v^\alpha \bar{v}^\beta z^a \bar{z}^b$ in (3.1). Now we Taylor expand the Hamiltonian with respect to $w$ and $z$ at the point $w = I$ and $z = 0$ respectively.

**Definition 3.1** (Degree decomposition)**.** Let $I$ be fixed as in (1.9). For every integer $d \geq -2$ and any regular Hamiltonian $H \in \mathcal{H}_{r,p}$ we define the following projection:

$$\Pi^d H := H^{(d)} := \sum_{\substack{m,\alpha,\beta,\delta\in\mathbb{N}^{\mathcal{S}},a,b\in\mathbb{N}^{\mathcal{S}^c} \\ \alpha\cap\beta=\emptyset\,,\ \delta\preceq m \\ 2|\delta|+|a|+|b|=d+2}} H_{m,\alpha,\beta,a,b} \binom{m}{\delta} I^{m-\delta} (|v|^2 - I)^\delta v^\alpha \bar{v}^\beta z^a \bar{z}^b \,. \tag{3.2}$$

where $\delta \preceq m$ means that $\delta_j \leq m_j$ for any $j \in \mathbb{Z}$.

In this way, if $\mathcal{S} = \mathbb{Z}$, projections coincide with the ones of section 4 of [9], while if $\mathcal{S} = \emptyset$, $H^{(d)}$ represents the usual homogeneous degree at $z = 0$.
In this way, given $H \in \mathcal{H}_{r,p}$, then

$$H = H^{(\leq 0)} + H^{(\geq 1)} \equiv H^{(-2)} + H^{(-1)} + H^{(0)} + H^{(\geq 1)} \tag{3.3}$$

where $H^{(-2)}$ consists of terms which are constant w.r.t. both $z$ and and the "auxiliary action" $w = |v|^2$, $H^{(-1)}$ is independent of the action but linear in the $z_j$, while $H^{(0)}$ contributes with two terms: the one linear in the action and independent of $z$, the second one quadratic in $z$ and independent of the action. Finally, $H^{(\geq 1)}$ is what is left and $X_{H^{(\geq 1)}}$ vanishes on $\mathcal{T}_I$.

The operators $\Pi^d$ define continuous projections satisfying $\Pi^d\Pi^d = \Pi^d$ and $\Pi^{d'}\Pi^d = \Pi^d\Pi^{d'} = 0$ for every $d' \neq d$, $d' \geq -2$. Moreover, this decomposition enjoys all the crucial properties required for a KAM scheme to converge, in particular they behave well with respect to Poisson brackets, that is:

$$\forall F, G \in \mathcal{H}_{r,p} \quad \left\{F, G^{\geq 1}\right\}^{(-2)} = 0$$

and

$$F^{(-2)} = 0 \implies \left\{F, G^{\geq 1}\right\}^{(-2)} = 0, \quad \text{and} \quad F^{(-1)} = 0 = F^{(-2)} \implies \left\{F, G^{\geq 1}\right\}^{(\leq 0)} = 0.$$

**Definition 3.2** (Normal Forms)**.** Let $I, r, p$ be as in (1.9). Let $D : \mathcal{Q} \to \mathbb{R}$ the linear map defined as

$$D(\omega) := \sum_{j\in\mathbb{Z}} \omega_j |u_j|^2 \,.$$

We will say that a Hamiltonian $N$ is in normal form at $\mathcal{T}_I$ with frequency $\omega$ if $N - D(\omega) \in \mathcal{H}_{r,p}^{(\geq 1)}$. We denote this the affine space by $\mathcal{N}_{r,p}(\omega; I)$.

**Theorem 3.** *Consider $r_0, p_0, \rho, r, \delta_0 > 0$ with $\rho \leq r_0/2$ and $r \leq r_0/2\sqrt{2}$. There exists $\bar{\epsilon}, \bar{C} > 0$, depending only on $\rho/r_0$ and $\delta_0$ such that the following holds. Let $\sqrt{I} \in \bar{B}_r(\mathtt{w}_{p_0+\delta_0})$ be such that $I_j = 0$, $\forall j \notin \mathcal{S}$. Consider a*

[7] Consisting in a reordering of the indexes $j$.

*family of normal forms* $N_0(\omega; I) \in \mathcal{N}_{r_0,p_0}(\omega; I)$*. Finally consider a Lipschitz family of Hamiltonians* $H(\omega)$*, with* $\omega \in D_{\gamma,S}$*, assume that* $H(\omega) - D(\omega) \in \mathcal{H}_{r_0,p_0}$ *and*

$$(1+\Theta)^4 \epsilon \leq \bar{\epsilon}\,, \qquad \text{where} \qquad \epsilon := \gamma^{-1} \|H - N_0\|_{r_0,p_0}\,, \quad \Theta = \gamma^{-1} \|D - N_0\|_{r_0,p_0}\,. \tag{3.4}$$

*Then there exist a symplectic diffeomorphism* $\Psi : B_{r_0-\rho}(\mathtt{w}_{p_0+\delta_0}) \to B_{r_0}(\mathtt{w}_{p_0+\delta_0})$*, close to the identity, a unique correction (counter term)* $\Lambda = \sum_j \lambda_j \left(|u_j|^2 - I_j\right)$*, Lipschitz depending on* $\omega \in \mathtt{D}_{\gamma,\mathcal{S}}$*, namely*

$$|\lambda(\omega)|_\infty + \gamma \frac{|\lambda(\omega) - \lambda(\omega')|_\infty}{|\omega - \omega'|_\infty} \leq \bar{C}\gamma(1+\Theta)^2\epsilon\,, \qquad \forall\, \omega, \omega' \in \mathtt{D}_{\gamma,\mathcal{S}}, \quad \omega \neq \omega' \tag{3.5}$$

*and a family of normal forms* $N(\omega; I) \in \mathcal{N}_{r_0-\rho,p_0+\delta_0}(\omega; I)$*, such that*

$$(\Lambda + H) \circ \Psi = N. \tag{3.6}$$

Theorem 1 follows from Theorem 3 in a straightforward way. One first rewrites $H_V$ in (1.5) as $D + P + \Lambda$ by setting $\lambda_j = j^2 - \omega_j + V_j$ so that it fits the hypotheses with $N_0 = D$, $\Theta = 0$ and $\epsilon \sim \varepsilon$ (recalling (1.8) and taking $\varepsilon_*$ small enough). Then Theorem 3 gives us the desired change of variables provided that $\Lambda$ is fixed in terms of the frequency $\omega$. Now we denote $\omega_j = \nu_j$ if $j \in \mathcal{S}$ and $\omega_j = \Omega_j$ otherwise. We get the equations

$$\begin{cases} \Omega_j + \lambda_j(\nu, \Omega) = j^2 + W_j\,, & \text{if } j \notin \mathcal{S} \\ \nu_j + \lambda_j(\nu, \Omega) = j^2 + V_j\,, & \text{if } j \in \mathcal{S}\,. \end{cases} \tag{3.7}$$

Now we Lipschitz extend the map $\lambda : \mathtt{D}_{\gamma,\mathcal{S}} \to \ell_\infty$ to the whole square $\mathcal{Q}$ and, by (3.5) and the Contraction Lemma, we solve the first equation finding $\Omega = \Omega(\nu)$. Finally we solve the second equation by setting $V_j = \nu_j + \lambda_j(\nu, \Omega(\nu)) - j^2$ for $j \in \mathcal{S}$. This concludes the proof of Theorem 1 and also shows (1.13).

The proof of Theorem 3 is based on an iterative scheme that kills out the obstructing terms, namely terms belonging to $\mathcal{H}^{(-2)}_{r,p}, \mathcal{H}^{(-1)}_{r,p}$ and $\mathcal{H}^{(0)}_{r,p}$, by solving homological equations of the form

$$L_\omega F^{(d)} = G^{(d)}, \qquad G^{(d)} \in \mathcal{H}^{(d)}_{r,p}, \quad d = -2, -1, 0. \tag{3.8}$$

where

$$L_\omega[\cdot] := \{\sum_j \omega_j |u_j|^2, \cdot\}\,, \qquad L_\omega H = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{N}^\mathbb{Z}} \omega \cdot (\boldsymbol{\alpha} - \boldsymbol{\beta}) H_{\boldsymbol{\alpha},\boldsymbol{\beta}} u^{\boldsymbol{\alpha}} \bar{u}^{\boldsymbol{\beta}}\,.$$

The convergence of the iterative KAM scheme comes from a good control of the solution $L_\omega^{-1} G^{(d)}$.

To explain this let us give some definitions. As is standard we denote by $\mathcal{H}^{\mathcal{K}}_{r,p}$ the kernel of $L_\omega$ and set

$$\Pi^{\mathcal{K}} H := \sum_{\boldsymbol{\alpha} \in \mathbb{N}^\mathbb{Z}} H_{\boldsymbol{\alpha},\boldsymbol{\alpha}} |u|^{2\boldsymbol{\alpha}}\,, \qquad \Pi^{\mathcal{R}} H := H - \Pi^{\mathcal{K}} H\,. \tag{3.9}$$

Correspondingly, we define the following subspaces of $\mathcal{H}_{r,p}$:

$$\mathcal{H}^{\mathcal{K}}_{r,p} := \{H \in \mathcal{H}_{r,p} : \quad \Pi^{\mathcal{K}} H = H\}\,, \qquad \mathcal{H}^{\mathcal{R}}_{r,p} := \{H \in \mathcal{H}_{r,p} : \quad \Pi^{\mathcal{R}} H = H\}\,. \tag{3.10}$$

these projections are continuous on $\mathcal{H}_{r,p}$.

On the subspace $\mathcal{H}^{\mathcal{R}}_{r,p}$, the Lie derivative operator $L_\omega$ is formally invertible with inverse

$$L_\omega^{-1} f := \sum \frac{f_{\boldsymbol{\alpha},\boldsymbol{\beta}}}{\mathrm{i}(\omega \cdot (\boldsymbol{\alpha} - \boldsymbol{\beta}))} u^{\boldsymbol{\alpha}} \bar{u}^{\boldsymbol{\beta}} \tag{3.11}$$

A good bound for the solutions of the homological equations (3.8) is a consequence of the following crucial

**Lemma 3.1** (Straightening the tangential dynamics)**.** *Fix* $0 < \gamma \le 1$ *and* $\mathcal{S}$ *as in Definition 1.1. Let* $\mathtt{D}_{\gamma,\mathcal{S}} \ni \omega \mapsto f(\omega) \in \mathcal{H}^{\le 2,\mathcal{R}}_{r,p}$ *be a Lipschitz map. Then for every* $0 < \delta < 1$ *we have*

$$\|L_\omega^{-1} f\|_{p+\delta,r} \le \frac{1}{\gamma} \exp\left( \mathtt{c} \exp\left( \left(\frac{2}{\delta}\right)^{1/\eta} \right) \right) \|H\|_{p,r}\,, \tag{3.12}$$

*for some constant* $\mathtt{c} = \mathtt{c}(i_*)$*, where* $i_*$ *was introduced in Definition 1.1.*

*Remark* 3.1. Estimates (3.12) improves when the set $\mathcal{S}$ is "more sparse". For example if $s(i) = 2^i$ one obtains

$$\|L_\omega^{-1} f\|_{r,p+\delta} \le c\gamma^{-1} e^{\frac{c}{\delta}\ln^2(\frac{1}{\delta})} \|f\|_{r,p}\,.$$

The proof of Lemma 3.1 is the real core of our result. It is simple if $f$ is supported only on $\mathcal{S}$ or $\mathcal{S}^c$. The crucial point is to control the interaction between tangential and normal sites. The key ingredient is that we are only considering Hamiltonians that are at most quadratic in the normal variables, which in turn are supported on a sparse lattice (recall Remark 1.3). This result should be compared with the corresponding one in [7] Proposition 7.1 item (M). In the latter paper in order to control $L_\omega^{-1} f$ we cannot take any $\delta > 0$ but instead must require $\delta \ge \tau_1$, where $\tau_1$ is some fixed quantity.

The proof of Theorem 3 is a superexponentially convergent KAM iterative scheme. We suitably choose the values of the parameter at each iterative step $j \in \mathbb{N}$, in particular the loss of regularity $\delta_j$ must be summable, e.g. $\delta_j \sim j^{-c}$, for some $c > 1$ and we need $c < \eta$. Indeed the divergence due to small divisors, which is of order $\exp(\exp(j^{c/\eta}))$ by (3.12), must be compensated by the superexponential convergence $\exp(-\exp(Cj))$ given by the KAM quadratic scheme. This justifies the choice $\eta > 1$.

It remains to comment about the proof of Theorem 2. The main point is that we can impose the strong diophantine condition (1.12) still obtaining a positive measure set. The crucial issue is the structure of the set $\mathcal{S}$ and the fact that we only need to consider denominators with $\sum_{s\in\mathcal{S}^c} |\ell_s| \le 2$. Note that our diophantine condition becomes the usual one in the case $\sum_{s\in\mathcal{S}^c} |\ell_s| = 0$, by just renaming the indices $i = s(i)$. To deal with the remaining terms ($\ell$ not supported only on the tangential sites) we use the constants of motion and the dispersive nature of the equation ($\omega_k \sim k^2$).

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